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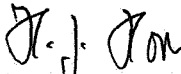
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16. ABSTRACT A model for trajectory computations for engineering-type application is described. The differential equations for the adjoint variables are derived and coded in FORTRAN. The program is written in a form to either take into account or neglect thrust, aerodynamic forces, planet rotation and oblateness, and altitude dependent winds.					
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DEFINITION OF SYMBOLS

A_e	Total exit area of thrustors
a	Velocity of sound
b	Mass flow of propulsion system
C	Constraint function
$C_{1,2,3}$	Coefficients of lift dependent drag polar terms [equation (16)]
C_D	Drag coefficient
C_{D0}	Drag coefficient for zero lift
C_L	Lift coefficient
f	Right-hand side of differential equations
G	Matrix $\partial f / \partial u$ with elements g_{kj} ; $k = 1, n$; $j = 1, nu$
GM	Planet mass * gravitational constant
H	Hamiltonian $H = \lambda_k f_k$
h	Altitude
$H0...H6$ $HI1...HI6$	Abbreviations [equations (21) through (28)]
H_{uu}	Second derivative matrix of Hamiltonian with respect to the controls
J	Geopotential coefficient
$l_{1,2,3}$	Transformation expression [equation (8)]
Ma	Mach number
m	Vehicle mass (ratio to initial mass)
n	Number of state variables
nu	Number of controls

DEFINITION OF SYMBOLS (Continued)

$n_{1,2,3}$	Transformation expression [equations (8)]
n_{zmax}	Maximum allowed normal acceleration due to lift
p_{∞}	Ambient atmospheric pressure
q	$0.5\rho V_a^2$ dynamic pressure
R_0	Planet radius (spherical approximation)
$\Delta R_0(\Lambda)$	Nonspherical radial component
r	Distance from planet mass center
S	Aerodynamic reference area
SWT	Thrust switching function
T	Thrust
t	Time
U	Radius normal inertial velocity component
U_e	Exit velocity of thrusters
u	Controls $u^T = (C_L, \mu, \sigma, \epsilon, b)$; $nu = 5$
V	Velocity
W	Wind velocity (indexed)
w	Square root expression for extremal thrust (primer) = [equations (22), (33)]
w_1	Square root expression for extremal thrust [equation (34)]
w_2	Square root expression for extremal bank angle [equation (35)]
X	State variable $X^T = (U, \chi_i, \dot{r}, r, \Lambda, m, \theta)$
x,y,z	Coordinates, forces in coordinate direction

DEFINITION OF SYMBOLS (Continued)

Greek Symbols

α	Angle of attack
β	Angle to determine extremal bank angle μ
γ	Flight path angle
ϵ	Thrust angle normal to vertical plane through vehicle center of gravity containing U
ϕ	Angle between vernal equinox and zero-meridian
χ_i	Azimuth angle (inertial) measured from north over east
$\Delta\chi_a$	Aerodynamic-inertial horizontal misalignment angle
Λ	Geocentric latitude
λ	Adjoint variable, Lagrangian multiplier
μ	Aerodynamic bank angle
ν	Constant Lagrangian multiplier for given end conditions
ρ	Air density
σ	Thrust angle from horizontal in vertical plane
$\varphi \tau$	Angles to determine extremal thrust direction
θ	Longitude
ω_p	Planet angular velocity

Indices

A	Atmospheric
a	Aerodynamic

DEFINITION OF SYMBOLS (Concluded)

c	Constraint
EW	East-west component
f	At final time
g	Radius normal (geodetic) coordinate system
h	Horizontal component
i	Inertial
i, j, k	Running indices
NS	North-south component
p	Planet (fixed)
rp	Index instead of \dot{r}
T	Thrust
u	Partial derivative with respect to the controls u
v	Vertical

Adjoints are marked with the corresponding state variable symbol.

$$\dot{()}\overset{\Delta}{=}\frac{d}{dt}()$$

TECHNICAL MEMORANDUM

VARIATIONAL DIFFERENTIAL EQUATIONS FOR ENGINEERING TYPE TRAJECTORIES CLOSE TO A PLANET WITH AN ATMOSPHERE

INTRODUCTION

This report documents the theory behind a subroutine for optimal trajectory computation. The mechanics involved are considered to be standard knowledge in atmospheric flight mechanics; therefore, no derivation of the equations of motion will be given. The subroutine is intended for use in connection with extremal field methods. Since a large amount of computational work load stems from the nonalignment of the inertial and the aerodynamic velocity vector, a special path in the subroutine has been provided for crude first investigations with aligned velocity vectors (no planet rotation, no wind).

The subroutine is intended for engineering-type applications. This determines the degree of sophistication for the mathematical model which is described in the next section (Mathematical Model). In the computation of optimal trajectories, the selection of coordinate systems in which the motion is to be described is even more crucial than it is for straightforward numerical integration. It affects not only the computational work load (e. g., use of trigonometric functions) and the integration accuracy but also the range of validity of linear approximations used in the iteration process and the interpretability of the adjoint variables. The coordinate systems chosen are described in the third section (coordinate Systems).

Symbols chosen in the report agree, in general, with those coded. The code is intended to be mnemonic (which, of course, is strongly biased by the author's background).

MATHEMATICAL MODEL

Experience with reentry trajectory computations [1, 2, 3] led to the definition of the following mathematical model.

Planetary Model

Only the gravitational force of the central body is considered. The J2-gravitational potential spherical harmonic term is taken into account in the vertical gravity component. Other influences of the planets' gravitational field or of moons, other planets, and the sun are disregarded.

The geometrical form of the planet is a flattened sphere. The actual shape is approximated by a third-order polynomial fit to the curve $\Delta R_0(\Lambda) =$ radial deviation from a sphere as function of the geocentric latitude for $0 \leq \Lambda \leq \pi/2$. Its derivative for $\Lambda = 0$ and $\pi/2$ is 0. For the earth the radius difference between pole and equator is approximately 21 km with a maximum in the slope between sphere and ellipsoid of almost 0.2 deg around $\Lambda = \pi/4$. Therefore, for shallow entry angles this has to be taken into account [2].

Atmospheric characteristics are a function of altitude alone. Density, pressure, and velocity-of-sound profiles have to be furnished by subroutines which also provide their derivatives. These profiles are considered to be independent of longitude, latitude, and time in the ascent or reentry region traversed.

Vehicle Model

The vehicle is considered to be a point mass. It has no moments of inertia, and its angular motions are considered to be instantaneously changeable controls.

There acts no aerodynamic side force. The vehicle is considered to have a plane of symmetry in which the aerodynamic force vector is bound to lie (no sideslip).

The aerodynamic characteristics of the vehicle are to be furnished by a subroutine. The program is set up for a drag polar type of representation ($C_D = C_{D_0} + C_1 C_L + C_2 C_L^2 + C_3 C_L^3$). Provision is made to take into account Mach-number-and altitude-(viscous interaction) dependent changes of the characteristics. Note that this does not allow for double-valued polars (back-side of the lift-drag curve). This case may be dealt with by taking the drag coefficient as control instead of the lift coefficient or by going to a parameter representation (angle of attack), which, however, doubles the amount of computation necessary.

Thrust is provided by rocket engines with constant mass flow. The effect of atmospheric back pressure is taken into account.

The thrust angle as control is considered to be unconstrained and not linked to a certain body axis. This yields information which might be taken into account for the design of the vehicle. Thrust angle constraints may be easily added.

COORDINATE SYSTEMS

The basic plane of reference is the equatorial plane of the planet.

Inertial Coordinate System (Index i)

A Cartesian planet-centered coordinate system with fixed inertial directions is considered to be an inertial system, neglecting the influence of the planets motion around the sun or largescale motions. The x_i - and y_i -axes lie in the equatorial plane, and the planet rotates around the z_i axis (Fig. 1). The vernal equinox is taken as the reference direction for x_i .

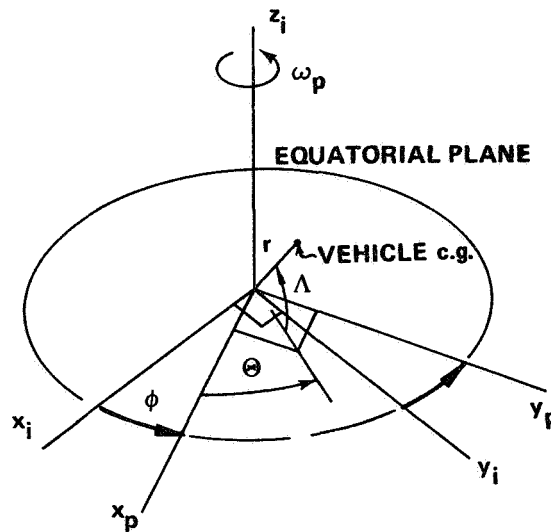


Figure 1. Inertial and planet fixed coordinate systems.

Planet-Fixed Coordinate System

This system, indexed p, rotates relative to the inertial system around the common z-axis. Its reference direction is taken to be the zero-meridian crossing with the equator. In this system the position of the vehicle is measured in polar coordinates: longitude θ , latitude Λ , and radius r . The altitude above sea level is computed from r and Λ . The rate of change of altitude is not equal to \dot{r} as it is for a spherical planet.

Radius Normal Coordinate System (Index g)

The equations of motion are written in a Cartesian coordinate system the z-axis of which is aligned with the radius vector from the planet mass-center to the vehicle center of gravity. The x-axis is in the direction of the inertial velocity component normal to the radius vector (Fig. 2). By this definition, there is no velocity component in the y-direction. Forces normal to the momentary plane of motion containing the planet mass center turn the "horizontal" component U of the velocity vector and thereby the orientation of the x-axis. The direction of the x-axis is measured by the inertial azimuth angle χ_i from north positive over east.

This choice of coordinate system is a compromise with respect to the factors mentioned in the introduction. For the most common reentry and the

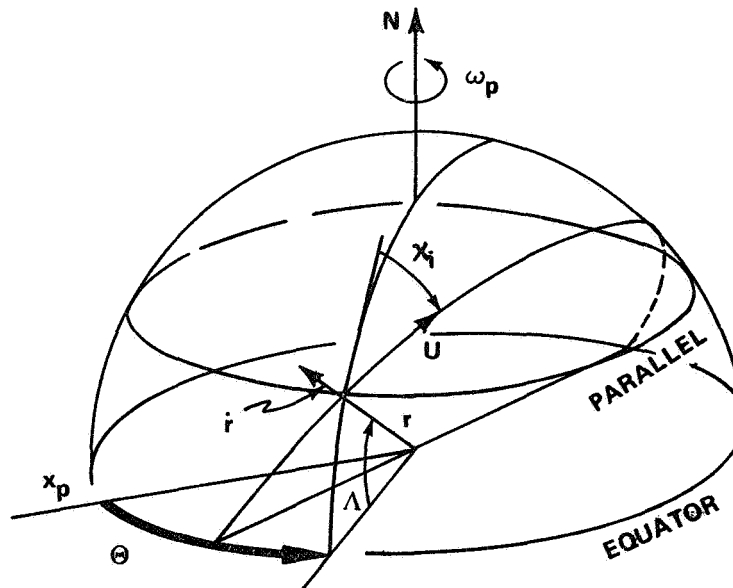


Figure 2. Radius normal coordinate system.

Aerodynamic Coordinate System (Index a)

EQUATIONS OF MOTION AND HAMILTONIAN FUNCTION

5

Transformation Aerodynamic-Radius Normal

From Figure 3 it is seen that the north-south and east-west aerodynamic velocity components are given by

$$V_{NS} = U \cos \chi_i + W_{NS} \quad (1)$$

$$V_{EW} = U \sin \chi_i + W_{EW} - r \omega_p \cos \Lambda, \quad (2)$$

where the wind components W are positive for north wind and east wind. The total horizontal aerodynamic velocity component is

$$\begin{aligned} V_{ah} = \sqrt{V_{NS}^2 + V_{EW}^2} = & \left\{ U^2 + 2U[\cos \chi_i W_{NS} \right. \\ & + \sin \chi_i (W_{EW} - r \omega_p \cos \Lambda)] + W_{NS}^2 \\ & \left. + (W_{EW} - r \omega_p \cos \Lambda)^2 \right\}^{1/2}. \end{aligned} \quad (3)$$

The vertical aerodynamic velocity component with W_v positive for a downwind is

$$V_v = \dot{r} + W_v \quad (4)$$

yielding a total velocity vector

$$V_a = (V_{NS}^2 + V_{EW}^2 + V_v^2)^{1/2}. \quad (5)$$

The case of vertical winds will not be pursued further ($W_v \equiv 0$). The aerodynamic flight path angle is given by

$$\sin \gamma_a = V_v / V_a \quad \text{or} \quad \tan \gamma_a = V_v / V_{ah}. \quad (6)$$

From Figure 3 one reads

$$\sin (\chi_i + \Delta \chi_a) = V_{EW} / V_{ah}$$

$$\cos (\chi_i + \Delta \chi_a) = V_{NS} / V_{ah}$$

which by trigonometric relations yields

$$\begin{aligned}\sin \Delta\chi_a &= (V_{EW} \cos \chi_i - V_{NS} \sin \chi_i)/V_{ah} = F(U, \chi_i, r, \Lambda) \\ \cos \Delta\chi_a &= (V_{EW} \sin \chi_i + V_{NS} \cos \chi_i)/V_{ah} = F(U, \chi_i, r, \Lambda)\end{aligned}\quad (7)$$

for the horizontal misalignment angle $\Delta\chi_a$. These expressions depend on the state variables U, χ_i, r, Λ as indicated after the second equal sign.

Forces are to be transformed from the aerodynamic into the radius normal coordinate system by

$$\begin{bmatrix} X_g \\ Y_g \\ Z_g \end{bmatrix} = \begin{bmatrix} l_1 & n_1 \\ l_2 & n_2 \\ l_3 & n_3 \end{bmatrix} \begin{bmatrix} X_a \\ Z_a \end{bmatrix}, \quad (8)$$

where

$$\begin{aligned}l_1 &= \cos \gamma_a \cos \Delta\chi_a, \\ l_2 &= \cos \gamma_a \sin \Delta\chi_a, \\ l_3 &= -\sin \gamma_a, \\ n_1 &= \cos \mu \sin \gamma_a \cos \Delta\chi_a + \sin \mu \sin \Delta\chi_a, \\ n_2 &= \cos \mu \sin \gamma_a \sin \Delta\chi_a - \sin \mu \cos \Delta\chi_a, \\ n_3 &= \cos \mu \cos \gamma_a.\end{aligned}$$

The bank angle μ is a control to be determined such that a pay-off function is extremized.

State Differential Equations

Aerodynamic and thrust terms¹ are grouped together:

$$\begin{aligned} \dot{X}_1 = \dot{U} = & \underbrace{-\frac{Ur}{r}} + \underbrace{\frac{T}{m} \cos \epsilon \cos \sigma} \\ & \underbrace{-\frac{Sq}{m} [C_D \cos \gamma_a \cos \Delta \chi_a]} \\ & \underbrace{+ C_L (\cos \mu \sin \gamma_a \cos \Delta \chi_a + \sin \mu \sin \Delta \chi_a)}] \quad , \quad (9) \end{aligned}$$

$$\begin{aligned} \dot{X}_2 = \dot{\chi}_i = & \underbrace{\frac{U}{r} \sin \chi_i \tan \Lambda} + \underbrace{\frac{T}{mU} \sin \epsilon} \\ & \underbrace{-\frac{Sq}{Um} [C_D \cos \gamma_a \sin \Delta \chi_a]} \\ & \underbrace{+ C_L (\cos \mu \sin \gamma_a \sin \Delta \chi_a - \sin \mu \cos \Delta \chi_a)}] \quad , \quad (10) \end{aligned}$$

$$\begin{aligned} \dot{X}_3 = \dot{\ddot{r}} = & \underbrace{\frac{U^2}{r} - \frac{GM}{r^2} \left[1 - J \left(\frac{R_0}{r} \right)^2 (3 \sin^2 \Lambda - 1) \right]} + \underbrace{\frac{T}{m} \cos \epsilon \sin \sigma} \\ & \underbrace{-\frac{Sq}{m} [C_D \sin \gamma_a - C_L \cos \gamma_a \cos \mu]} \quad , \quad (11) \end{aligned}$$

$$\dot{X}_4 = \dot{r} = X_2 \quad , \quad (12)$$

$$\dot{X}_5 = \dot{\Lambda} = \underbrace{\frac{U}{r} \cos \chi_i} \quad , \quad (13)$$

$$\dot{X}_6 = \dot{m} = \underline{\underline{-b}} \quad , \quad (14)$$

$$\dot{X}_7 = \dot{\theta} = \underbrace{\frac{U \sin \chi_i}{r \cos \Lambda}} - \omega_p \quad . \quad (15)$$

¹Aerodynamic terms will be designated by broken underlining, thrust terms by double underlining, and inertial terms by single underlining.

The lift coefficient C_L is the second control. The drag coefficient C_D is dependent on Mach-Reynolds number and the lift coefficient. A representation of the form

$$C_D = C_{D_0} + C_1 C_L + C_2 C_L^2 + C_3 C_L^3 \quad (16)$$

is implemented, where C_{D_0} and the C_i contain the influence of the Mach number or the combined Mach-Reynolds number effects (viscous interaction). This form is adapted to a bivariate spline-representation of the drag polar. The term S is the aerodynamic reference area. The dynamic pressure

$$q = \frac{1}{2} \rho V_a^2 \quad (17)$$

contains five state variables [see eqs. (1) through (5)] which are U , \dot{r} , r , χ_i , Λ , where r and Λ also determine the air density $\rho(h)$. The altitude h above the planet reference surface is written

$$h = r - R_0 + a_i \Lambda_i, \quad i = 1, 3 \quad (18)$$

with R_0 = equivalent spherical planet radius (6371.2 km for earth). The thrust T contains an atmospheric term

$$T = b \cdot U_e - (p_\infty \cdot A_e) \quad (19)$$

where

U_e = exit velocity ,

p_∞ = free-stream pressure ,

A_e = exit area of engines .

It is decomposed into a component in the vertical plane containing X_g (thrust angle σ positive from horizontal upward) and a component normal to this plane (angle ϵ , positive for increasing χ_i).

The Hamiltonian Function

The adjoint variables are symbolized by the state variable symbol as index (rp for \dot{r}). The abbreviations used in the derivation are also used as code in the FORTRAN program. The Hamiltonian is decomposed into inertial, thrust, and atmospheric terms:

$$H = \underline{H_I} + \underline{H_T} + \underline{H_A} \quad . \quad (20)$$

Inertial terms:

$$\begin{aligned} H_I = \frac{U}{r} (H_4) - \lambda_{rp} \frac{GM}{r^2} \left[1 - J \left(\frac{R_0}{r} \right)^2 (3 \sin^2 \Lambda - 1) \right] \\ + \lambda_{\dot{r}} \dot{r} - \lambda_{\theta} \omega_p \quad , \end{aligned} \quad (21)$$

where

$$H_4 = -(HI2) + \sin \chi_i (HI5) + \lambda_{\Lambda} \cos \chi_i \quad ,$$

$$HI2 = \lambda_{\dot{u}} \dot{r} - \lambda_{rp} U \quad ,$$

$$HI5 = \lambda_{\chi} \tan \Lambda + \lambda_{\theta} / \cos \Lambda$$

Thrust terms:

$$H_T = b \left(\frac{U_a}{m} w - \lambda_m \right) - \frac{p_{\infty} A_e}{m} w \quad , \quad (22)$$

where

$$w = w_1 \cos \epsilon + \frac{\lambda_{\chi}}{U} \sin \epsilon \quad ,$$

$$w_1 = \lambda_u \cos \sigma + \lambda_{rp} \sin \sigma \quad .$$

In order to determine the optimal thrust program, H_T has to be extremized with respect to the mass flow b and the thrust angles σ and ϵ . The last term in equation (22) is dependent on the atmosphere too.

Atmospheric terms:

$$H_A = \frac{p_\infty A_e}{m} w - \frac{Sq}{m} [C_D H1 + C_L (H2 \cos \mu + H3 \sin \mu)] \quad , \quad (23)$$

where

$$H1 = H0 \cos \gamma_a + \lambda_{rp} \sin \gamma_a \quad ,$$

$$H2 = H0 \sin \gamma_a - \lambda_{rp} \cos \gamma_a \quad ,$$

$$H3 = \lambda_u \sin \Delta\chi_a - \frac{\lambda_\chi}{U} \cos \Delta\chi_a \quad ,$$

and

$$H0 = \lambda_u \cos \Delta\chi_a + \lambda_\chi / U \cdot \sin \Delta\chi_a \quad .$$

The term in square brackets is symbolized by

$$H6 = C_D H1 + C_L (H2 \cos \mu + H3 \sin \mu) \quad . \quad (24)$$

The extremization of $H6$ with respect to the controls C_L (or C_D , α) and the bank angle μ yields candidates for the optimal controls. The first term in equation (23) is an atmosphere-dependent thrust term.

Reduced Equations for Aligned Inertial and Aerodynamic Velocity Vectors

The misalignment angle $\Delta\chi_a$ is identically zero, and aerodynamic and inertial flight path angles are identical:

$$\gamma_a = \gamma_i = \tan^{-1}(\dot{r}/U) \quad . \quad (25)$$

With

$$V_i = (\dot{r}^2 + U^2)^{1/2} \quad , \quad (26)$$

there is

$$\sin \gamma_a = \dot{r}/V_i \quad ; \quad \cos \gamma_a = U/V_i \quad , \quad (27)$$

and the atmosphere-related part of the Hamiltonian may be written

$$\begin{aligned} H'_A = & -\frac{Sq}{mV_i} [C_D HI1 + C_L (\cos \mu HI2 + \sin \mu HI3)] \\ & - \frac{p_\infty A_e}{m} w \quad , \end{aligned} \quad (28)$$

where

$$HI1 = U\lambda_u + \dot{r}\lambda_{rp} \quad ,$$

$$HI2 = \dot{r}\lambda_u - U\lambda_{rp} \quad ,$$

$$HI3 = -\lambda_\chi V_i/U \quad .$$

This concludes the presentation of the basic equations for the determination of the optimal control and the derivation of the adjoint differential equations.

EXTREMAL CONTROL

In this section, applying the maximum principle, the extremal control will be derived for both maximization and minimization of a pay off function (i. e., minimization and maximization of the Hamiltonian with respect to the controls).

Thrust Control

Thrust Magnitude. Since the thrust is linear in the mass flow rate b , the expression in parentheses in equation (22) acts as a thrust switching function

$$SWT = \frac{U_e}{m} w - \lambda_m \quad . \quad (29)$$

The extremal value for b is given in Table 1.

TABLE 1. EXTREMAL FLOW RATE

Value of SWT	Pay Off Maximization $\min_b H$	Minimization $\max_b H$
> 0	$b = 0$	$b = b_{\max}$
$= 0$	indeterminate (singular-arc)	
< 0	$b = b_{\max}$	$b = 0$

The case of a singular arc will not be considered here.

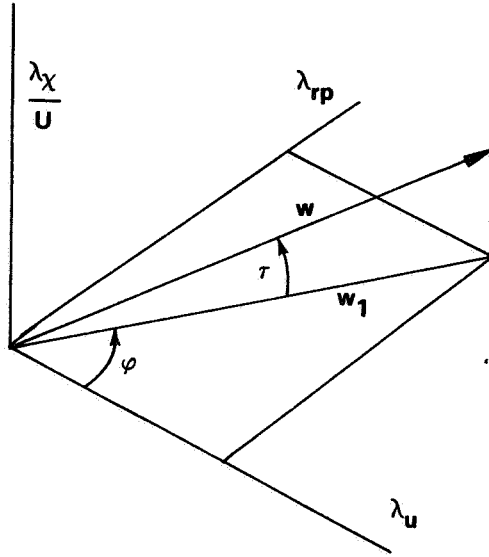


Figure 4. Extremal thrust direction and primer vector.

Thrust Direction. Define the two vectors (Fig. 4)

$$w_1' = (\lambda_u^2 + \lambda_{rp}^2)^{1/2} \quad (30)$$

and

$$w' = [w_1'^2 + (\lambda_\chi/U)^2]^{1/2} ; \quad (31)$$

then the following relations hold:

$$\begin{aligned} \lambda_u &= w_1' \cos \varphi , \\ \lambda_{rp} &= w_1' \sin \varphi , \\ w_1' &= w' \cos \tau , \\ \lambda_\chi/U &= w' \sin \tau . \end{aligned} \quad (32)$$

Introducing these relations into the second and third equations of equation (22) yields

$$w = w' (\cos \tau \cos \epsilon + \sin \tau \sin \epsilon) = w' \cos (\tau - \epsilon) \quad (33)$$

and

$$w_1 = w_1' (\cos \varphi \cos \sigma + \sin \varphi \sin \sigma) = w_1' \cos (\varphi - \sigma) . \quad (34)$$

The term w_1 has its maximum value w_1' for $\sigma = \varphi$ and its minimum $-w_1'$ for $\sigma = \pi + \varphi$. w , known as primer vector, has its maximum w' for $\epsilon = \tau$ and its minimum $-w'$ for $\epsilon = \pi + \tau$.

The extremal controls σ in the vertical plane and ϵ normal to it, together with the trigonometric expressions in which they appear in the differential equations, are given in Table 2.

TABLE 2. EXTREMAL THRUST DIRECTION

Case Extremal Values	Pay Off	
	Maximization $\min_{\sigma, \epsilon} H$	Minimization $\max_{\sigma, \epsilon} H$
$\sigma =$	$\pi + \tan^{-1}(\lambda_{rp}/\lambda_u)$	$\tan^{-1}(\lambda_{rp}/\lambda_u)$
$\epsilon =$	$\pi + \tan^{-1}[\lambda_\chi/(Uw_1)]$	$\tan^{-1}[\lambda_\chi/(Uw_1)]$
$\cos \epsilon \sin \sigma =$	$\lambda_{rp}/(-w)$	λ_{rp}/w
$\cos \epsilon \cos \sigma =$	$\lambda_u/(-w)$	λ_u/w
$\sin \epsilon =$	$\lambda_\chi/(-Uw)$	$\lambda_\chi/(Uw)$
Note: The terms w_1 and w are taken from equations (30) and (31).		

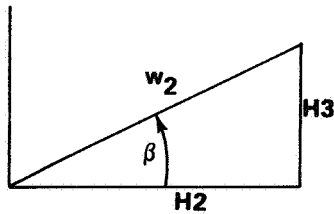


Figure 5. Determination of extremal bank angle.

Control of the Aerodynamic Forces

Introduce (Fig. 5)

$$w_2 = \left[(H2)^2 + (H3)^2 \right]^{1/2} ; \quad (35)$$

then

$$\beta = \tan^{-1}(H3/H2) , \quad (36)$$

and the lift-dependent part of the Hamiltonian in equation (23) may be written

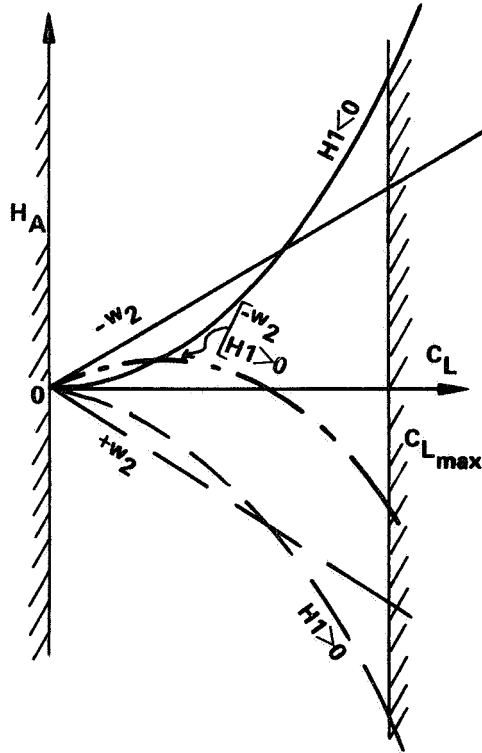
$$H'_A = -C_D H_1 - C_L w_2 (\cos \beta \cos \mu + \sin \beta \sin \mu). \quad (37)$$

For positive lift coefficients $C_L > 0$, the Hamiltonian is maximal with respect to μ for $\cos(\beta - \mu) = -1$; i. e., $\mu = \pi + \beta$, minimal for $\mu = \beta$.

The lift coefficient will be confined to the range

$$0 \leq C_L \leq C_{L_{\max}}. \quad (38)$$

Downward accelerating aerodynamic forces may be generated by bank angles $|\beta| > \pi/2$.



Lift coefficients off the boundaries are determined from

$$\frac{\partial H_A}{\partial C_L} = -\frac{\partial C_D}{\partial C_L} H_1 - (\pm w_2) = 0. \quad (39)$$

For a drag polar of the form in equation (16), there follows

$$C_{L_{\text{opt}}} = \frac{C_2}{3C_3} \left[\left(1 + \frac{6C_3}{C_2} C_{L_2} \right)^{1/2} - 1 \right] \quad (40)$$

with

$$C_{L_2} = - \left(\pm \frac{w_2}{H_1} + C_1 \right) / (2C_2) \quad (41)$$

as optimal control for $C_3 = 0$.

From Figure 6 it is seen that C_L can only be off the boundary when $w_2 \cdot H_1 < 0$;

Figure 6. Determination of extremal lift coefficient from the components of H_A [see eq. (37)].

i. e., when they have different signs. $C_L = C_{L_{\max}}$ for $w_2, H1 \leq 0$ and maximization of the Hamiltonian or $w_2, H1 \geq 0$ and min. H. $C_L = 0$ for $w_2, H1 \leq 0$ and minimization or $w_2, H1 \geq 0$ and maximization of the Hamiltonian. The case $C_L = C_{L_{\max}}$ may also occur for $w_2 \cdot H1 < 0$.

Table 3 summarizes the extremal control for the aerodynamic forces.

TABLE 3. EXTREMAL CONTROL OF AERODYNAMIC FORCES

Control \ Case		Pay Off	
		Maximization $\min_{C_L, \mu} H$	Minimization $\max_{C_L, \mu} H$
$\mu =$		$\tan^{-1}(H3/H2)$	$\pi + \tan^{-1}(H3/H2)$
$\sin \mu =$		$H3/w_2$	$H3/(-w_2)$
$\cos \mu =$		$H2/w_2$	$H2/(-w_2)$
$C_L =$			
1.	$w_2, H1 \leq 0$	0	$C_{L_{\max}}$
2.	$w_2, H1 \geq 0$	$C_{L_{\max}}$	0
3.	$w_2, H1 = 0$	singular	singular
4.	$w_2 \cdot H1 < 0$	Eqs. (40) and (41) with $+ w_2$	Eqs. (40) and (41) with $- w_2$

ADJOINT EQUATIONS

The adjoint variables λ used in eqs. (20) to (28) to form the Hamiltonian function and in the preceding section to obtain the extremal control are given by the differential equations

$$\dot{\lambda} = -\frac{\partial H}{\partial X} \quad . \quad (42)$$

In this section the additive terms for the inertial, thrusting, and atmospheric flight parts are derived. These summands of the right-hand side of the differential equations will be marked by the indices I, T, A. How many of these terms are computed and added up is controlled by logical variables from the main program. If all terms are used, we have

$$\dot{\lambda} = \dot{\lambda}_{\underline{I}} + \dot{\lambda}_{\underline{T}} + \dot{\lambda}_{\underline{A}} \quad . \quad (43)$$

Inertial Terms

From equations (21) and (42) follows:

$$\dot{\lambda}_{u, I} = -\left[(H4) + U\lambda_{rp}\right]/r \quad , \quad (44)$$

$$\dot{\lambda}_{rp, I} = U\lambda_u/r - \lambda_r \quad , \quad (45)$$

$$\dot{\lambda}_{r, I} = \left\{ (H4)U - 2\lambda_{rp} \frac{GM}{r} \left[1 - 2J\left(\frac{R_0}{r}\right)^2 (3 \sin^2 \Lambda - 1) \right] \right\} / r^2 \quad (46)$$

$$\dot{\lambda}_{\chi, I} = \frac{U}{r} (\lambda_{\Lambda} \sin \chi_i - (H15) \cos \chi_i) \quad , \quad (47)$$

$$\begin{aligned} \dot{\lambda}_{\Lambda, I} = & -\frac{U}{r} \frac{\sin \chi}{\cos^2 \Lambda} (\lambda_{\chi} + \lambda_{\theta} \sin \Lambda) \\ & - \lambda_{rp} J \frac{GM \cdot R_0^2}{r^4} 6 \sin \Lambda \cos \Lambda \quad , \end{aligned} \quad (48)$$

$$\dot{\lambda}_{m, I} = 0 \quad , \quad (49)$$

$$\dot{\lambda}_{\theta, I} = 0 \quad . \quad (50)$$

Thrust Terms

From eq. (22) we have for vacuum thrust

$$\dot{\lambda}_{U, T} = b \frac{U_e}{m} \frac{\lambda}{U} \frac{\sin \epsilon}{U} \quad (51)$$

$$\dot{\lambda}_{m, T} = b \frac{U_e}{m^2} (\pm w) \quad (52)$$

Atmospheric Terms

Thrust Terms. From the second term in equation (22), we have

$$\dot{\lambda}_{r, AT} = \pm w \frac{A_e}{m} \frac{\partial p_\infty}{\partial h} \frac{\partial h}{\partial r} \quad ; \quad \frac{\partial h}{\partial r} = 1 \quad , \quad (53)$$

$$\dot{\lambda}_{\Lambda, AT} = \pm w \frac{A_e}{m} \frac{\partial p_\infty}{\partial h} \frac{\partial h}{\partial \Lambda} \quad , \quad (54)$$

$$\dot{\lambda}_{m, AT} = \mp w \frac{A_e p_\infty}{m^2} \quad (55)$$

There are two sets of adjoint equations for the atmospheric part, one for aligned inertial and atmospheric velocity vectors, which is relatively simple and will be treated first, and one for the general case.

Aligned Velocity Vectors. Equations (25) to (28) yield

$$\begin{aligned} \text{HI6} \quad &= C_D \underbrace{(U \lambda_u + \dot{r} \lambda_{rp})}_{\text{HI1}} \\ &+ C_L \left[\cos \mu \underbrace{(\lambda_u \dot{r} - \lambda_{rp} U)}_{\text{HI2}} - \sin \mu \lambda_\chi / \cos \gamma \right] \end{aligned} \quad (56)$$

and

$$\begin{aligned} \frac{\partial (\text{HI6})}{\partial U} &= C_D \lambda_u + \frac{\partial C_D}{\partial \text{Ma}} \frac{\partial \text{Ma}}{\partial U} \quad (\text{HI1}) \\ &\quad - C_L \left(\lambda_{rp} \cos \mu - \frac{\lambda}{V_i} \tan^2 \gamma \sin \mu \right) , \end{aligned} \quad (57)$$

$$\begin{aligned} \frac{\partial (\text{HI6})}{\partial \dot{r}} &= C_D \lambda_{rp} + \frac{\partial C_D}{\partial \text{Ma}} \frac{\partial \text{Ma}}{\partial \dot{r}} \quad (\text{HI1}) \\ &\quad + C_L \left(\lambda_u \cos \mu - \frac{\lambda}{V_i} \tan \gamma \sin \mu \right) , \end{aligned} \quad (58)$$

$$\frac{\partial (\text{HI6})}{\partial r} = \left(\frac{\partial C_D}{\partial \text{Ma}} \frac{\partial \text{Ma}}{\partial h} + \frac{\partial C_D}{\partial h} \right) (\text{HI1}) , \quad (59)$$

$$\frac{\partial (\text{HI6})}{\partial \Lambda} = \frac{\partial (\text{HI6})}{\partial r} \frac{\partial h}{\partial \Lambda} , \quad (60)$$

$$\frac{\partial V_i}{\partial U} = \cos \gamma ; \quad \frac{\partial V_i}{\partial \dot{r}} = \sin \gamma . \quad (61)$$

For the additive terms in the adjointed differential equation according to equation (42)

$$\dot{\lambda}_{u, A} = \frac{S\rho}{2m} \left[\cos \gamma (\text{HI6}) + V_i \cdot \partial (\text{HI6}) / \partial U \right] , \quad (62)$$

$$\dot{\lambda}_{rp, A} = \frac{S\rho}{2m} \left[\sin \gamma (\text{HI6}) + V_i \cdot \partial (\text{HI6}) / \partial \dot{r} \right] , \quad (63)$$

$$\dot{\lambda}_{r, A} = \frac{SV_i}{2m} \left[\frac{\partial \rho}{\partial h} (\text{HI6}) + \rho \cdot \partial (\text{HI6}) / \partial r \right] , \quad (64)$$

$$\dot{\lambda}_{\Lambda, A} = \frac{SV_i}{2m} \left[\frac{\partial \rho}{\partial h} \frac{\partial h}{\partial \Lambda} (\text{HI6}) + \rho \cdot \partial (\text{HI6}) / \partial \Lambda \right] , \quad (65)$$

$$\dot{\lambda}_{m, A} = -\frac{S\rho V_i}{2m^2} \text{ (HI6)} \quad . \quad (66)$$

General Case. Since the misalignment angles and the aerodynamic velocity vector are relatively complicated functions of the state variables, the partial derivatives of the abbreviations given in equations (23) and (24) are determined in the appendix. Up to equation (A-49) no adjoint variables are involved. The equations (A-50) to (A-56) have to be computed for each set of adjoint equations.

The aerodynamic force terms in equation (23) may be written as

$$\frac{S}{2m} \cdot \rho(r, \Lambda) \cdot V_a^2(U, \dot{r}, \chi_i, r, \Lambda) \cdot H6(U, \dot{r}, \chi_i, r, \Lambda) \quad .$$

From this, the additive terms in the adjoint differential equations to U , \dot{r} , and χ_i are

$$\dot{\lambda}_{X, A} = \frac{S\rho V_a}{2m} \left[2 \frac{\partial V_a}{\partial X} H6 + V_a \frac{\partial H6}{\partial X} \right] ; \quad (67)$$

to r and Λ we have

$$\dot{\lambda}_{X, A} = \frac{SV_a}{2m} \left[\left(\frac{\partial \rho}{\partial X} (H6) + \rho \frac{\partial H6}{\partial X} \right) V_a + 2\rho (H6) \frac{\partial V_a}{\partial X} \right] . \quad (68)$$

Finally, there is

$$\dot{\lambda}_{m, A} = -\frac{S\rho}{2m^2} V_a^2 (H6) \quad . \quad (69)$$

TERMS USED IN GRADIENT METHODS

In a general form, equations (9) to (15) are written

$$\dot{X} = f(x, u, t) \quad . \quad (70)$$

The control u is eliminated in extremal field optimization methods by applying the calculus of variation or the maximum principle using

Lagrangian multipliers (see preceding sections, Extremal Control and Adjoint Equations). In gradient methods, the iteration centers around an estimated time history for the controls. The gradient H_u needed in these algorithms and the time-varying weighting matrix H_{uu}^{-1} used in the min-H method [4] will be given in this section.

With

$$X^T = (U, \chi_i, \dot{r}, r, \Lambda, m, \theta)$$

and the control vector

$$u^T = (C_L, \mu, \sigma, \epsilon, b) \quad , \quad (71)$$

the matrix $G = f_u$ has the following form:

$$G = \begin{bmatrix} g_{11} & g_{12} & g_{13} & g_{14} & g_{15} \\ g_{21} & g_{22} & 0 & g_{24} & g_{25} \\ g_{31} & g_{32} & g_{33} & g_{34} & g_{35} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad , \quad (72a)$$

where

$$g_{11} = -\frac{Sq}{m} \left[\cos \Delta\chi_a \left(\cos \gamma_a \frac{\partial C_D}{\partial C_L} + \sin \gamma_a \cos \mu \right) + \sin \Delta\chi_a \sin \mu \right] \quad ,$$

$$g_{12} = -\frac{Sq}{m} C_L (\cos \mu \sin \Delta\chi_a - \sin \mu \sin \gamma_a \cos \Delta\chi_a) \quad ,$$

$$g_{13} = \frac{-T}{m} \cos \epsilon \sin \sigma \quad ,$$

$$g_{14} = -\frac{T}{m} \sin \epsilon \cos \sigma \quad ,$$

$$g_{15} = \frac{U_a}{m} \cos \epsilon \cos \sigma \quad ,$$

$$g_{21} = -\frac{Sq}{Um} \left[\left(\cos \gamma_a \frac{\partial C_D}{\partial C_L} + \sin \gamma_a \cos \mu \right) \sin \Delta \chi_a \right. \\ \left. - \cos \Delta \chi_a \sin \mu \right] \quad ,$$

$$g_{22} = \frac{Sq}{Um} C_L (\sin \mu \sin \gamma_a \sin \Delta \chi_a + \cos \mu \cos \Delta \chi_a) \quad ,$$

$$g_{24} = \frac{T}{mU} \cos \epsilon \quad ,$$

$$g_{45} = \frac{U_e}{mU} \sin \epsilon \quad ,$$

$$g_{31} = -\frac{Sq}{m} \left(\sin \gamma_a \frac{\partial C_D}{\partial C_L} - \cos \gamma_a \cos \mu \right) \quad ,$$

$$g_{32} = -\frac{Sq}{m} C_L \cos \gamma_a \sin \mu \quad ,$$

$$g_{33} = \frac{T}{m} \cos \epsilon \cos \sigma \quad ,$$

$$g_{34} = -\frac{T}{m} \sin \epsilon \sin \sigma \quad ,$$

$$g_{35} = \frac{U_a}{m} \cos \epsilon \sin \sigma \quad . \tag{72b}$$

From equations (22) and (23), the following relation for H_u is obtained:

$$H_u^T = \begin{bmatrix} -\frac{Sq}{m} \left[H1 \cdot \frac{\partial C_D}{\partial C_L} + H2 \cos \mu + H3 \sin \mu \right] \\ -\frac{Sq}{m} C_L (H3 \cos \mu - H2 \sin \mu) \\ \frac{T}{m} \cos \epsilon (\lambda_{rp} \cos \sigma - \lambda_u \sin \sigma) \\ \frac{T}{m} \left(\frac{\lambda_\chi}{U} \cos \epsilon - w_1 \sin \epsilon \right) \\ \frac{U}{m} w - \lambda_m \end{bmatrix} \quad (73)$$

which yields for H_{uu}

$$H_{uu} = \begin{bmatrix} h_{11} & h_{21} & 0 & 0 & | & 0 \\ h_{21} & h_{22} & 0 & 0 & | & 0 \\ 0 & 0 & h_{33} & h_{34} & | & h_{35} \\ 0 & 0 & h_{43} & h_{44} & | & h_{45} \\ \hline 0 & 0 & h_{53} & h_{54} & & 0 \end{bmatrix} \quad (74)$$

with

$$h_{11} = -\frac{Sq}{m} (H1) \frac{\partial^2 C_D}{\partial C_L^2} ,$$

$$h_{21} = h_{12} = -\frac{Sq}{m} (H3 \cos \mu - H2 \sin \mu) ,$$

$$h_{22} = \frac{Sq}{m} C_L (H2 \cos \mu + H3 \sin \mu) ,$$

$$h_{33} = -\frac{T}{m} \cos \epsilon \, w_1 \quad ,$$

$$h_{34} = h_{43} = \frac{T}{m} \sin \epsilon \, (\lambda_u \sin \sigma - \lambda_{rp} \cos \sigma) \quad ,$$

$$h_{35} = h_{53} = -\frac{U}{m} \cos \epsilon \, (\lambda_u \sin \sigma - \lambda_{rp} \cos \sigma) \quad ,$$

$$h_{45} = h_{54} = \frac{U}{m} \left(\frac{\lambda}{U} \cos \epsilon - w_1 \sin \epsilon \right) \quad .$$

It is seen that thrust and aerodynamic forces are completely decoupled. Since the Hamiltonian is linear in the mass flow rate b , its influence may be accounted for by changes in the switching times τ_e and τ_l and the fifth row and column of H_{uu} can be deleted.

For the computation of the influence functions [5], the expressions $G^T \lambda$ are needed which are similar in structure to H_u but are formed with the adjoint variables for one given final condition; where as, H_u is formed with the Lagrangian multipliers as a superposition of the adjoints [4]

$$\lambda = \lambda_\phi + \nu_i \lambda_{\Psi i} \quad (75)$$

with ν_i as constant multipliers for the given final conditions $\Psi_i(X_f, t_f) = 0$.

CONTROL DEPENDENT STATE SPACE CONSTRAINTS

If the trajectory is bound to lie in a certain region of the state space, it is in general made up of arcs on the state space boundary and arcs off it. The case where the state space constraint function involves one of the controls directly is of special interest in atmospheric flight mechanics since both normal acceleration and reradiative heating constraints are of this type.

In this section the necessary relations for applying the constraint optimization technique of reference [6] are derived for the above-mentioned constraints.

Reradiative Heating Constraint

It is assumed that a quasisteady heating equilibrium model is a good enough approximation to the dynamic process. Then the constraint function may be written in the form

$$C = h(r, \Lambda) - h_L \left[C_L, V_a(U, \chi_i, \dot{r}, r, \Lambda) \right] \geq 0, \quad (76)$$

where h_L is a limit altitude above which the vehicle has to stay for a given velocity and lift coefficient in order not to exceed the heating limit. On a boundary arc, the lift coefficient as the only control involved in the constraint directly has to be determined by $C = 0$. Then, in order to satisfy the necessary conditions of optimality, additive terms in the adjoint differential equations have to be evaluated (second summand):

$$\dot{\lambda} = - \frac{\partial f}{\partial X} \lambda + \left[G \left(\frac{\partial C}{\partial u} \right)^{-1} \frac{\partial C}{\partial X} \right]^T \lambda. \quad (77)$$

From equation (76) follows

$$\frac{\partial C}{\partial u} = \begin{bmatrix} \frac{\partial h_L}{\partial u} & 0 & 0 & 0 & 0 \end{bmatrix} \quad (78)$$

$$\frac{\partial C}{\partial X} = - \frac{\partial h_L}{\partial V_a} \left[\frac{\partial V_a}{\partial U} \frac{\partial V_a}{\partial \chi_i} \frac{\partial V_a}{\partial \dot{r}} \left(\frac{\partial V_a}{\partial r} - \frac{\partial h / \partial r}{\partial h_L / \partial V_a} \right) \left(\frac{\partial V_a}{\partial \Lambda} - \frac{\partial h / \partial \Lambda}{\partial h_L / \partial V_a} \right) \right]. \quad (79)$$

Together with equation (72), the second summand in equation (77) may be written

$$\lambda_{u, c_1} = \frac{\partial h_L}{\partial V_a} (\lambda_u g_{11} + \lambda_\chi g_{21} + \lambda_{rp} g_{31}) \frac{\partial V_a}{\partial U} \bigg/ \frac{\partial h_L}{\partial C_L}$$

and with the abbreviations in equation (23)

$$\lambda_{u, c_1} = \frac{Sq}{m} \frac{\partial C_L}{\partial V_a} \frac{\partial V_a}{\partial U}. \quad (H7) \quad (80)$$

where

$$H7 = \frac{\partial C_D}{\partial C_L} (H1) + \cos \mu (H2) + \sin \mu (H3) \quad . \quad (81)$$

In extremal field methods, the last two terms sum up to w_2 [see equations (35) to (37)]. For the other adjoints, one obtains similarly

$$\lambda_{\chi, C_1} = -\frac{Sq}{m} \frac{\partial C_L}{\partial V_a} \frac{\partial V_a}{\partial \chi_i} (H7) \quad , \quad (82)$$

$$\lambda_{rp, C_1} = -\frac{Sq}{m} \frac{\partial C_L}{\partial V_a} \frac{\partial V_a}{\partial \dot{r}} (H7) \quad , \quad (83)$$

$$\lambda_{r, C_1} = \frac{Sq}{m} \left(\frac{\partial C_L}{\partial h_L} - \frac{\partial C_L}{\partial V_a} \frac{\partial V_a}{\partial r} \right) \cdot H7 \quad , \quad (84)$$

$$\lambda_{\Lambda, C_1} = \frac{Sq}{m} \left(\frac{\partial h/\partial \Lambda}{\partial h_L/\partial C_L} - \frac{\partial C_L}{\partial V_a} \frac{\partial V_a}{\partial \Lambda} \right) \cdot H7 \quad . \quad (85)$$

The derivatives $\partial C_L/\partial V_a$ and $\partial C_L/\partial h_L$ have to be furnished by the subroutine which generates the control on the boundary $C_L(\rho, V_a)$.

Normal Acceleration Constraint

The constraint equation is

$$C = \frac{S}{2mg_0} \rho V_a^2 C_L - n_{zmax} \leq 0 \quad . \quad (86)$$

From

$$\frac{\partial C}{\partial u} = \begin{bmatrix} \frac{S\rho V_a^2}{2mg_0} & 0 & 0 & 0 & 0 \end{bmatrix} \quad (87)$$

and

$$\frac{\partial C}{\partial X} = \frac{SV_a \rho C_L}{gm_0} \left[\frac{\partial V_a}{\partial u} \frac{\partial V_a}{\partial \chi_i} \frac{\partial V_a}{\partial \dot{r}} \left(\frac{\partial V_a}{\partial r} + \frac{V_a}{2\rho} \frac{\partial \rho}{\partial r} \right) \left(\frac{\partial V_a}{\partial \Lambda} + \frac{V_a}{2\rho} \frac{\partial \rho}{\partial \Lambda} \right) \right] , \quad (88)$$

the additive terms in equation (77) become

$$\begin{aligned} \lambda_{u, C_2} &= -0.5 S \rho V_a C_L \frac{\partial V_a}{\partial U} \quad (H7) , \\ \lambda_{\chi, C_2} &= -0.5 SV_a \rho C_L \frac{\partial V_a}{\partial \chi_i} \quad (H7) , \\ \lambda_{rp, C_2} &= -0.5 SV_a \rho C_L \frac{\partial V_a}{\partial \dot{r}} \quad (H7) , \\ \lambda_{r, C_2} &= -0.5 SV_a \rho C_L \left(\frac{\partial V_a}{\partial r} + \frac{V_a}{2\rho} \frac{\partial \rho}{\partial r} \right) \cdot H7 , \\ \lambda_{\Lambda, C_2} &= -0.5 SV_a \rho C_L \left(\frac{\partial V_a}{\partial \Lambda} + \frac{V_a}{2\rho} \frac{\partial \rho}{\partial \Lambda} \right) H7 . \end{aligned} \quad (89)$$

On the boundary, the lift coefficient is determined from

$$C_L = 2mg_0 n_{zmax} / (S \rho V_a^2) . \quad (90)$$

Note that for $C_L \geq C_{L_{min}}$ and $C_{L_{min}} = 0$, this constraint can always be

satisfied, whereas the heating constraint above may lead to a reduced set of entry conditions onto the constraint for feasible trajectories. This difference comes from the fact that the normal acceleration is generated by the control, while the heating constraint is a state space constraint where the control enters as a parameter.

FORTRAN SUBROUTINE

The mathematical model described in the previous sections has been coded in FORTRAN for use in an extremal field algorithm. The program is self-explanatory once the logical parameters are described, which is the purpose of the following section.

Controlling Logic

The purpose of incorporating ten logical variables was to achieve a flexible and efficient subroutine. Five of the control variables are input parameters into the main program and remain constant throughout the iterations. They partly specify the case to be run.

TABLE 4. LOGICAL CONTROL PARAMETERS
FOR CASE SPECIFICATION

Logical Variable	State	Result (deg)
Lobl	. True.	Planet oblateness taken into account: altitude $h = r - R_0 + \Delta h(\Lambda)$, equation (18)
	. False.	Spherical planet $h = r - R_0$
Lrot	. True.	Planet rotation taken into account
	. False.	Nonrotating planet ($\omega_p = 0$)
LWI	. True.	Horizontal winds $W(h)$ taken into account
	. False.	No winds
LDVA	. True.	= Lrot .or. LWI
	. False.	Special path for atmospheric terms to reduce work load in the case of aligned inertial and aerodynamic velocity vectors
LMIMAX	. True.	Payoff maximization
	. False.	Payoff minimization

The other five logical variables are stored in a two-dimensional array and controlled by a switching function subroutine which sets them as a function of the state and adjoint variables during the iteration. The first index refers to the trajectory section under consideration, the second to the number of the switching function. Their effect is given in Table 5.*

TABLE 5. LOGICAL PARAMETERS
SET BY SWITCHING FUNCTIONS

Logical Variable	State	Result
LU(J, 1)	. True.	Atmospheric terms are taken into account
	. False.	Atmospheric terms are by-passed (e. g., for $h > 100$ km)
LU(J, 2)	. True.	Thrust is on
	. False.	All thrust terms are by-passed
LU(J, 3)	. True.	Vehicle flies on heating constraint boundary
	. False.	Heating constraint not effective
LU(J, 4)	. True.	Vehicle flies on normal acceleration boundary
	. False.	Normal acceleration constraint not effective
LU(J, 5)	. True.	Flight with $C_{L_{max}}$
	. False.	C_L determined from \underline{X} and $\underline{\lambda}$

* The FORTRAN-program is documented under separate cover. Copies are available from: Trajectory & Optimization Theory Branch, Aero-Astroynamics Laboratory, Marshall Space Flight Center, Alabama 35812, Attn: Mr. J. R. Redus, S&E-Aero-GT.

APPENDIX

DERIVATION OF PARTIAL DERIVATIVES NEEDED TO FORM THE ADJOINT DIFFERENTIAL EQUATIONS

For nonaligned inertial and aerodynamic velocity vectors, the following dependencies are noted:

$$V_{NS} \left\{ U, \chi_i, W_{NS} [h(r, \Lambda)] \right\} = V_{NS} (U, \chi_i, r, \Lambda) \quad ,$$

$$V_{EW} \left\{ U, \chi_i, r, \Lambda, W_{EW} [h(r, \Lambda)] \right\} = V_{EW} (U, \chi_i, r, \Lambda) \quad ,$$

$$V_{ah} (V_{EW}, V_{NS}) = V_{ah} (U, \chi_i, r, \Lambda) \quad ,$$

$$V_a (V_{ah}, \dot{r}) = V_a (U, \dot{r}, \chi_i, r, \Lambda) \quad ,$$

$$\gamma_a (V_{ah}, V_a) = \gamma_a (U, \dot{r}, \chi_i, r, \Lambda) \quad ,$$

$$\Delta \chi_a (V_{EW}, V_{NS}, V_{ah}, \chi_i) = \Delta \chi_a (U, \chi_i, r, \Lambda) \quad ,$$

$$\rho [h(r, \Lambda)] = \rho (r, \Lambda) \quad ,$$

$$p_\infty [h(r, \Lambda)] = p_\infty (r, \Lambda) \quad ,$$

$$Ma [V_a, h(r, \Lambda)] = Ma (U, \dot{r}, \chi_i, r, \Lambda) \quad ,$$

$$C_D (Ma, h) = C_D (U, \dot{r}, \chi_i, r, \Lambda) \quad .$$

The equations are given in the section, Equations of Motion and Hamiltonian Function. From these the following partial derivatives are obtained:

North-south aerodynamic velocity component:

$$V_{NS} = U \cos \chi_i + W_{NS} \quad . \tag{A-1}$$

$$\begin{aligned}\frac{\partial V_{NS}}{\partial U} &= \cos \chi_i \quad ; \quad \frac{\partial V_{NS}}{\partial \chi_i} = -U \sin \chi_i \\ \frac{\partial V_{NS}}{\partial r} &= \frac{\partial W_{NS}}{\partial h} \quad ; \quad \frac{\partial V_{NS}}{\partial \Lambda} = \frac{\partial W_{NS}}{\partial h} \frac{\partial h}{\partial \Lambda} \quad .\end{aligned}\tag{A-2}$$

East-west aerodynamic velocity component:

$$V_{EW} = U \sin \chi_i + W_{EW} - \omega_p r \cos \Lambda \quad .\tag{A-3}$$

$$\begin{aligned}\frac{\partial V_{EW}}{\partial U} &= \sin \chi_i \quad ; \quad \frac{\partial V_{EW}}{\partial \chi_i} = U \cos \chi_i \\ \frac{\partial V_{EW}}{\partial r} &= \frac{\partial W_{EW}}{\partial h} - \omega_p \cos \Lambda \quad ; \quad \frac{\partial V_{EW}}{\partial \Lambda} = \frac{\partial W_{EW}}{\partial h} \frac{\partial h}{\partial \Lambda} + \omega_p r \sin \Lambda \quad .\end{aligned}\tag{A-4}$$

Radius normal aerodynamic velocity component:

$$V_{ah} = \left(V_{EW}^2 + V_{NS}^2 \right)^{1/2} \quad ,\tag{A-5}$$

$$\frac{\partial V_{ah}}{\partial U} = \left(V_{NS} \frac{\partial V_{NS}}{\partial U} + V_{EW} \frac{\partial V_{EW}}{\partial U} \right) / V_{ah} = \cos \Delta \chi_a \quad ,\tag{A-6}$$

$$\frac{\partial V_{ah}}{\partial \chi_i} = \left(V_{NS} \frac{\partial V_{NS}}{\partial \chi_i} + V_{EW} \frac{\partial V_{EW}}{\partial \chi_i} \right) / V_{ah} = U \sin \Delta \chi_a \quad ,\tag{A-7}$$

$$DVAHDR = \frac{\partial V_{ah}}{\partial r} = \left[V_{NS} \frac{\partial W_{NS}}{\partial h} + V_{EW} \left(\frac{\partial W_{EW}}{\partial h} - \omega_p \cos \Lambda \right) \right] / V_{ah} \quad ,\tag{A-8}$$

$$DVAHDL = \frac{\partial V_{ah}}{\partial \Lambda} = \left(V_{NS} \frac{\partial V_{NS}}{\partial \Lambda} + V_{EW} \frac{\partial V_{EW}}{\partial \Lambda} \right) / V_{ah} \quad .\tag{A-9}$$

Aerodynamic velocity:

$$V_a = \left(V_{ah}^2 + \dot{r}^2 \right)^{1/2} , \quad (A-10)$$

$$\frac{\partial V_a}{\partial U} = \frac{\partial V_{ah}}{\partial U} \frac{V_{ah}}{V_a} = \cos \gamma_a \cos \Delta \chi_a , \quad (A-11)$$

$$\frac{\partial V_a}{\partial \chi_i} = \frac{\partial V_{ah}}{\partial \chi_i} \frac{V_{ah}}{V_a} = U \cos \gamma_a \sin \Delta \chi_a , \quad (A-12)$$

$$\frac{\partial V_a}{\partial \dot{r}} = \dot{r}/V_a = \sin \gamma_a , \quad (A-13)$$

$$\frac{\partial V_a}{\partial r} = \cos \gamma_a \frac{\partial V_{ah}}{\partial r} ; \quad \frac{\partial V_a}{\partial \Lambda} = \cos \gamma_a \frac{\partial V_{ah}}{\partial \Lambda} . \quad (A-14)$$

Aerodynamic flight path angle:

$$\sin \gamma_a = \dot{r}/V_a , \quad (A-15)$$

$$DSGDU = \frac{\partial \sin \gamma_a}{\partial U} = \frac{-\dot{r}}{V_a^2} \frac{\partial V_a}{\partial U} = -\sin \gamma_a \cos \gamma_a \cos \Delta \chi_a / V_a , \quad (A-16)$$

$$DSGDRP = \frac{\partial \sin \gamma_a}{\partial \dot{r}} = \left(V_a - \dot{r} \frac{\partial V_a}{\partial \dot{r}} \right) / V_a^2 = \cos^2 \gamma_a / V_a , \quad (A-17)$$

$$DSGDC = \frac{\partial \sin \gamma_a}{\partial \chi_i} = \frac{-\dot{r}}{V_a^2} \frac{\partial V_a}{\partial \chi_i} = -\sin \gamma_a \cos \gamma_a \sin \Delta \chi_a U / V_a , \quad (A-18)$$

$$DSGDR = \frac{\partial \sin \gamma_a}{\partial r} = -\sin \gamma_a \cos \gamma_a \frac{\partial V_{ah}}{\partial r} / V_a , \quad (A-19)$$

$$DSGDLA = \frac{\partial \sin \gamma_a}{\partial \Lambda} = -\sin \gamma_a \cos \gamma_a \frac{\partial V_{ah}}{\partial \Lambda} / V_a , \quad (A-20)$$

$$\cos \gamma_a = V_{ah}/V_a, \quad (A-21)$$

$$\begin{aligned} DCGDU &= \frac{\partial \cos \gamma_a}{\partial U} = \left(V_a \frac{\partial V_{ah}}{\partial U} - V_{ah} \frac{\partial V_a}{\partial U} \right) / V_a^2 \\ &= \sin^2 \gamma_a \cos \Delta\chi_a / V_a, \end{aligned} \quad (A-22)$$

$$DCGDRP = \frac{\partial \cos \gamma_a}{\partial \dot{r}} = -\sin \gamma_a \cos \gamma_a / V_a, \quad (A-23)$$

$$DCGDC = \frac{\partial \cos \gamma_a}{\partial \chi_i} = \sin^2 \gamma_a \sin \Delta\chi_a U / V_a, \quad (A-24)$$

$$DCGDR = \frac{\partial \cos \gamma_a}{\partial r} = \frac{\partial V_{ah}}{\partial r} \sin^2 \gamma_a / V_a, \quad (A-25)$$

$$DCGDLA = \frac{\partial \cos \gamma_a}{\partial \Lambda} = \frac{\partial V_{ah}}{\partial \Lambda} \sin^2 \gamma_a / V_a. \quad (A-26)$$

Radius normal aerodynamic misalignment angle $\Delta\chi_a$:

$$\sin \Delta\chi_a = (V_{EW} \cos \chi_i - V_{NS} \sin \chi_i) / V_{ah}, \quad (A-27)$$

$$\begin{aligned} DSCDU &= \frac{\partial \sin \Delta\chi_a}{\partial U} = \left[\left(\frac{\partial V_{EW}}{\partial U} \cos \chi_i - \frac{\partial V_{NS}}{\partial U} \sin \chi_i \right) V_{ah} \right. \\ &\quad \left. - \sin \Delta\chi_a V_{ah} \frac{\partial V_{ah}}{\partial U} \right] / V_{ah}^2 \\ &= -\sin \Delta\chi_a \cos \Delta\chi_a / V_{ah}, \end{aligned} \quad (A-28)$$

$$\begin{aligned} DSCDC &= \frac{\partial \sin \Delta\chi_a}{\partial \chi_i} = \left[U(\cos^2 \chi_i + \sin^2 \chi_i) V_{ah} \right. \\ &\quad \left. - \sin \Delta\chi_a V_{ah} \frac{\partial V_{ah}}{\partial \chi_i} \right] / V_{ah}^2 = U \cos^2 \Delta\chi_a / V_{ah}, \end{aligned} \quad (A-29)$$

$$\begin{aligned} \text{DSCDR} &= \frac{\partial \sin \Delta\chi_a}{\partial r} = \left(\frac{\partial V_{EW}}{\partial r} \cos \chi_i - \frac{\partial V_{NS}}{\partial r} \sin \chi_i \right. \\ &\quad \left. - \sin \Delta\chi_a \frac{\partial V_{ah}}{\partial r} \right) / V_{ah} , \end{aligned} \quad (\text{A-30})$$

$$\begin{aligned} \text{DSCDLA} &= \frac{\partial \sin \Delta\chi_a}{\partial \Lambda} = \left(\frac{\partial V_{EW}}{\partial \Lambda} \cos \chi_i - \frac{\partial V_{NS}}{\partial \Lambda} \sin \chi_i \right. \\ &\quad \left. - \sin \Delta\chi_a \frac{\partial V_{ah}}{\partial \Lambda} \right) / V_{ah} , \end{aligned} \quad (\text{A-31})$$

$$\cos \Delta\chi_a = (V_{EW} \sin \chi_i + V_{NS} \cos \chi_i) / V_{ah} , \quad (\text{A-32})$$

$$\begin{aligned} \text{DCCDU} &= \frac{\partial \cos \Delta\chi_a}{\partial U} = \left[\left(\frac{\partial V_{EW}}{\partial U} \sin \chi_i + \frac{\partial V_{NS}}{\partial U} \cos \chi_i \right) \right. \\ &\quad \left. - \cos \Delta\chi_a \frac{\partial V_{ah}}{\partial U} \right] / V_{ah} = \sin^2 \Delta\chi_a / V_{ah} , \end{aligned} \quad (\text{A-33})$$

$$\begin{aligned} \text{DCCDC} &= \frac{\partial \cos \Delta\chi_a}{\partial \chi_i} = \left[U(\cos \chi_i \sin \chi_i - \sin \chi_i \cos \chi_i) \right. \\ &\quad \left. - \cos \Delta\chi_a \frac{\partial V_{ah}}{\partial \chi_i} \right] / V_{ah} \\ &= -U \sin \Delta\chi_a \cos \Delta\chi_a / V_{ah} , \end{aligned} \quad (\text{A-34})$$

$$\begin{aligned} \text{DCCDR} &= \frac{\partial \cos \Delta\chi_a}{\partial r} = \left(\frac{\partial V_{EW}}{\partial r} \sin \chi_i + \frac{\partial V_{NS}}{\partial r} \cos \chi_i \right. \\ &\quad \left. - \cos \Delta\chi_a \frac{\partial V_{ah}}{\partial r} \right) / V_{ah} , \end{aligned} \quad (\text{A-35})$$

$$\begin{aligned} \text{DCCDLA} = \frac{\partial \cos \Delta\chi_a}{\partial \Lambda} = & \left(\frac{\partial V_{EW}}{\partial \Lambda} \sin \chi_i + \frac{\partial V_{NS}}{\partial \Lambda} \cos \chi_i \right. \\ & \left. - \cos \Delta\chi_a \frac{\partial V_{ah}}{\partial \Lambda} \right) / V_{ah} . \end{aligned} \quad (\text{A-36})$$

Velocity of sound, air density and pressure a , ρ , $p_\infty = f[h(r, \Lambda)]$:

$$\frac{\partial \rho}{\partial r} = \frac{\partial \rho}{\partial h} \quad ; \quad \frac{\partial \rho}{\partial \Lambda} = \frac{\partial \rho}{\partial h} \frac{\partial h}{\partial \Lambda} \quad (\text{A-37})$$

$$\frac{\partial p}{\partial r} = \frac{\partial p}{\partial h} \quad ; \quad \frac{\partial p}{\partial \Lambda} = \frac{\partial p}{\partial h} \frac{\partial h}{\partial \Lambda} \quad (\text{A-38})$$

$$\frac{\partial a}{\partial r} = \frac{\partial a}{\partial h} \quad ; \quad \frac{\partial a}{\partial \Lambda} = \frac{\partial a}{\partial h} \frac{\partial h}{\partial \Lambda} . \quad (\text{A-39})$$

Mach number:

$$\text{Ma} = V_a / a , \quad (\text{A-40})$$

$$\text{DMADU} = \frac{\partial \text{Ma}}{\partial U} = \frac{\partial V_a}{\partial U} / a = \cos \gamma_a \cos \Delta\chi_a / a , \quad (\text{A-41})$$

$$\text{DMADRP} = \frac{\partial \text{Ma}}{\partial \dot{r}} = \frac{\partial V_a}{\partial \dot{r}} / a = \sin \gamma_a / a \quad (\text{A-42})$$

$$\text{DMADC} = \frac{\partial \text{Ma}}{\partial \chi_i} = \frac{\partial V_a}{\partial \chi_i} / a = \frac{U}{a} \cos \gamma_a \sin \Delta\chi_a , \quad (\text{A-43})$$

$$\text{DMADR} = \frac{\partial \text{Ma}}{\partial r} = \left(\frac{\partial V_a}{\partial r} - \text{Ma} \frac{\partial a}{\partial r} \right) / a , \quad (\text{A-44})$$

$$\text{DMADLA} = \frac{\partial \text{Ma}}{\partial \Lambda} = \left(\frac{\partial V_a}{\partial \Lambda} - \text{Ma} \frac{\partial a}{\partial \Lambda} \right) / a . \quad (\text{A-45})$$

Drag coefficient:

$$C_D = C_D(Ma, h) , \quad (A-46)$$

$$DCDDU = \frac{\partial C_D}{\partial U} = \frac{\partial C_D}{\partial Ma} \frac{\partial Ma}{\partial U} \text{ similar for } \dot{r}, \chi_i , \quad (A-47)$$

$$DCDDR = \frac{\partial C_D}{\partial r} = \frac{\partial C_D}{\partial Ma} \frac{\partial Ma}{\partial r} + \frac{\partial C_D}{\partial h} , \quad (A-48)$$

$$DCDDL\Lambda = \frac{\partial C_D}{\partial \Lambda} = \frac{\partial C_D}{\partial Ma} \frac{\partial Ma}{\partial \Lambda} + \frac{\partial C_D}{\partial h} \frac{\partial h}{\partial \Lambda} . \quad (A-49)$$

With these expressions the partial derivatives of the composite terms of equation (23) may be written:

$$DHODU = \frac{\partial H0}{\partial U} = \frac{\partial \cos \Delta\chi_a}{\partial U} \lambda_u + \lambda_\chi \left(\frac{\partial \sin \Delta\chi_a}{\partial U} - \sin \Delta\chi_a / U \right) / U . \quad (A-50)$$

Let X stand for χ_i, r, Λ ; then we have

$$\frac{\partial H0}{\partial X} = \lambda_u \frac{\partial \cos \Delta\chi_a}{\partial X} + \frac{\lambda_\chi}{U} \frac{\partial \sin \Delta\chi_a}{\partial X} \quad (A-51)$$

$$\frac{\partial H3}{\partial X} = \lambda_u \frac{\partial \sin \Delta\chi_a}{\partial X} - \frac{\lambda_\chi}{U} \frac{\partial \cos \Delta\chi_a}{\partial X} . \quad (A-52)$$

With X standing for U, $\dot{r}, \chi_i, r, \Lambda$, the partials of H1 and H2 may be written

$$\frac{\partial H1}{\partial X} = \frac{\partial H0}{\partial X} \cos \gamma_a + H0 \frac{\partial \cos \gamma_a}{\partial X} + \lambda_{rp} \frac{\partial \sin \gamma_a}{\partial X} , \quad (A-53)$$

$$\frac{\partial H2}{\partial X} = \frac{\partial H0}{\partial X} \sin \gamma_a + H0 \frac{\partial \sin \gamma_a}{\partial X} - \lambda_{rp} \frac{\partial \cos \gamma_a}{\partial X} , \quad (A-54)$$

$$\frac{\partial H3}{\partial U} = \lambda_u \frac{\partial \sin \Delta\chi_a}{\partial U} - \lambda_\chi \left(\frac{\partial \cos \Delta\chi_a}{\partial U} - \cos \Delta\chi_a / U \right) / U . \quad (A-55)$$

Finally we have as partials of H6 [see equation (24)]

$$\frac{\partial H6}{\partial X} = \frac{\partial C_D}{\partial X} H1 + C_D \frac{\partial H1}{\partial X} + C_L \cos \mu \frac{\partial H2}{\partial X} + C_L \sin \mu \frac{\partial H3}{\partial X} \quad .$$

(A-56)

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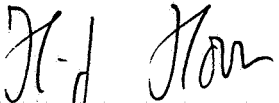
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VARIATIONAL DIFFERENTIAL EQUATIONS FOR ENGINEERING TYPE TRAJECTORIES CLOSE TO A PLANET WITH AN ATMOSPHERE

By E. D. Dickmanns

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